Math 201B: Selected Homework Solutions

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Problem. *Let (f_n) be a sequence of unit vectors in a Hilbert space. Prove that there is a weak-convergent subsequence.*

Proof. Let (e_n) be a basis for the Hilbert space H. Note that $|(f_n, e_1)| \leq \|f_n\| \|e_1\| = 1$, so we have a sequence of complex numbers $(f_n, e_1) \in \overline{\mathbb{D}}$. The closed unit disk is sequentially compact, so there is a subsequence (f_n^1) of (f_n) so that (f_n^1, e_1) converges to some complex number a_1. Inductively, if we have chosen a subsequence (f_n^k) of (f_n) so that the sequence (f_n^k, e_j) converges (to some a_j) for each $j = 1, \ldots, k$, then we can choose a subsequence (f_n^{k+1}) of (f_n^k) so that (f_n^{k+1}, e_j) converges for each $j = 1, \ldots, k+1$. This subsequence is chosen in the same way as (f_n^1).

Consider the 'diagonal' subsequence (f_n^n). By construction we have $(f_n^n, e_k) \to a_k$ for each $k \in \mathbb{N}$. Define $f = \sum_j a_j e_j$, so that $(f_n^n, e_k) \to (f, e_k)$ for each $k \in \mathbb{N}$. It remains to show that this series converges and that $(f_n^n - f, g) \to 0$ for each $g \in H$.

Fix a positive integer N and take n large enough that $|(f_n^n, e_k) - a_k| < 1/N$ for each $k = 1, 2, \ldots, N$. Then we have

$$\sum_{k=1}^{N} |a_k|^2 \leq \sum_{k=1}^{N} |(f_n^n, e_k) - a_k|^2 + \sum_{k=1}^{N} |(f_n^n, e_k)|^2 \leq 1 + \sum_{k=1}^{\infty} |(f_n^n, e_k)|^2 = 2.$$

Since N was arbitrary and the series $\sum |a_k|^2$ consists of positive terms, taking $N \to \infty$ gives $\|f\| = \sum_1^\infty |a_k|^2 < \infty$. Hence $f \in H$.

Let $g \in H$ and write $g = \sum_j b_j e_j$, where $\sum_j |b_j|^2 < \infty$. Fix $N > 0$ and write

$$\begin{aligned}
|(f_n^n - f, g)| &\leq \left| \left(f_n^n - f, \sum_{j=1}^{N} b_j e_j \right) \right| + \left| \left(f_n^n - f, \sum_{j=N+1}^{\infty} b_j e_j \right) \right| \\
&\leq \left| \left(f_n^n - f, \sum_{j=1}^{N} b_j e_j \right) \right| + \|f_n^n - f\| \left(\sum_{j=N+1}^{\infty} |b_j|^2 \right)^{1/2} \\
&\leq \left| \left(f_n^n - f, \sum_{j=1}^{N} b_j e_j \right) \right| + (1 + \|f\|) \left(\sum_{j=N+1}^{\infty} |b_j|^2 \right)^{1/2}.
\end{aligned}$$

Taking $n \to \infty$ gives

$$\limsup_{n \to \infty} |(f_n^n - f, g)| \leq (1 + \|f\|) \left(\sum_{j=N+1}^{\infty} |b_j|^2 \right)^{1/2}.$$

Since (b_j) is square summable and N was arbitrary, taking $N \to \infty$ gives $\lim (f_n^n - f, g) = 0$. ∎

Problem. *Let $H = L^2[0,1]$ and define $T : H \to H$ by $(Tf)(t) = tf(t)$. Show that T is bounded, symmetric, not compact, and has no eigenvectors.*

Proof. Let $f \in H$. Then

$$\|Tf\|_2^2 = \int_0^1 |tf(t)|^2 \, dt \le \int_0^1 |f(t)|^2 \, dt = \|f\|_2^2,$$

so T is bounded. If $g \in H$ as well, then

$$(Tf, g) = \int_0^1 tf(t)\overline{g(t)} \, dt = \int_0^1 f(t)\overline{tg(t)} \, dt = (f, Tg),$$

so $T = T^*$. (Remark: in fact, any self-adjoint operator is automatically bounded as well. This is a minor corollary of the closed-graph theorem). Suppose that $Tf = \lambda f$ for some $\lambda \in \mathbb{R}$ and $f \in H$. Then $(\lambda - t)f(t) = 0$ almost everywhere; that is, $f = 0$ a.e. and hence $f = 0$ in the (quotient space) $L^2[0,1]$. Therefore T has no eigenvectors. If T were compact it would have an eigenvector, so we conclude that T is not compact. □

Problem. *Let $(\phi_k)_{k=1}^\infty$ be an orthonormal basis for a Hilbert space H. Define $T : H \to H$ by $T(\phi_k) = (1/k)\phi_{k+1}$. Show that T is comapct and has no eigenvectors.*

Proof. Let $S : H \to H$ be the shift operator: $S : \phi_k \mapsto \phi_{k+1}$. If we define $A : \phi_k \mapsto (1/k)\phi_k$ as well, then $A, S \in \mathcal{L}(H)$ and $T = SA$. Notice that A is a diagonal operator with 'diagonal entries' (that is, eigenvalues) $\lambda_k = 1/k$. Since $\lambda_k \to 0$, we conclude that A is compact (as remarked in the text). Since the compact operators form an ideal (a theorem in text), we find that $T = SA$ is also compact.

Suppose that $Tf = \lambda f$ for some $\lambda \in \mathbb{C}$ and $f \in H$. We can write $f = \sum a_k \phi_k$ for some complex scalars (a_k). Then we find that

$$0 = (\lambda - T) \sum_{k=1}^\infty a_k \phi_k = \lambda a_1 \phi_1 + (\lambda a_2 - a_1)\, \phi_2 + \left(\lambda a_3 - \frac{a_2}{2}\right) \phi_3 + \cdots .$$

Each coefficient in this basis expansion must vanish. If $\lambda = 0$, then $0 = a_1 = a_2 = \cdots$. In this case, $f = 0$ and hence not an eigenvector. Suppose instead that $\lambda \neq 0$. Then $0 = a_1 = a_2 = a_3 = \cdots$ follows again. Either way, $f = 0$. We conclude that f has no eigenvectors. □